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General two-loop beta functions for gauge theories with arbitrary scalar fields

I Jack and H Osborn

University of Cambridge, Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, England

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Abstract. A renormalisation group β function is defined for the scalar potential and is calculated to two-loop order for general gauge scalar field theories.

Recently the two-loop counterterms for an arbitrary gauge scalar field theory have been computed using background field techniques and dimensional regularisation by van Damme (1982). Within our approach to such calculations (Jack and Osborn 1982), these results have been confirmed (Jack 1983). For any specific theory this information can be used to obtain the corresponding two-loop approximations to the β functions, for any of the coupling constants or mass terms that are relevant in renormalisation group equations. However, the choice of such coupling constant or mass parameters is somewhat arbitrary. In this paper we show how the results of van Damme (1982) can be used to determine for the entire scalar potential U a corresponding β function, to the two-loop level, without any particular choice of U. For renormalisable field theories $U(\varphi)$ is a general quartic polynomial in a set of scalar fields $\{\varphi_i\}$, as is also then the associated $\beta_U(\varphi)$. They may both be readily decomposed in terms of the coefficients of a convenient set of independent monomials in φ; for U these form the physical parameters of the theory, coupling constants and masses. Nevertheless the primary result for β_U is perhaps more straightforward and may possibly be of interest in the discussion of grand unified theories.

Regarding φ as a column vector, the general gauge–scalar field theory under consideration is described by a Lagrangian density

$$\mathscr{L} = -(4\mu^{\varepsilon})^{-1}(F_{\mu\nu}Z_A F^{\mu\nu}) + \tfrac{1}{2}(D_\mu\varphi)^{\mathrm{T}} Z_\varphi D^\mu\varphi - V(\varphi) \tag{1}$$

where

$$D_\mu = \partial_\mu + A^\varphi_\mu, \qquad A^\varphi_\mu = A_{\mu,a}T_a, \tag{2}$$

is the covariant derivative for the representation of the gauge group G defined by φ. $\{T_a\}$ are a set of antisymmetric matrix generators of G in this representation satisfying

$$[T_a, T_b] = c_{abc}T_c \tag{3}$$

and then $F^\varphi_{\mu\nu} = [D_\mu, D_\nu]$. The generators of the adjoint representation are $(t^{\mathrm{ad}}_a)_{bc} = -c_{abc}$ and for X, Y vectors belonging to this representation the invariant scalar product is $(XY) = X_aY_a$. For invariance under G

$$[t^{\mathrm{ad}}_a, Z_A] = 0, \qquad [T_a, Z_\varphi] = 0, \qquad V(\varphi) = V(g^\varphi\varphi). \tag{4}$$

Z_A, Z_φ and V are chosen so as to cancel the divergences that arise order by order in the loop expansion. Using dimensional regularisation, $\mathscr{L}$ is extended to d dimensions and a regularisation scale mass μ is introduced. If, for $\varepsilon = 4-d$,

$$\mu^{-\varepsilon} V_u(\mu^{\varepsilon/2}\varphi) = V(Z_\varphi^{-1/2}\varphi) \tag{5}$$

then the parameters in $V_u(\varphi)$, and from (1) Z_A also, retain their canonical dimensions even when $\varepsilon \neq 0$. To zeroth order we take

$$Z_A^{(0)} = N^{-1}, \qquad Z_\varphi^{(0)} = 1, \qquad V_u(\varphi)^{(0)} = U(\varphi), \tag{6}$$

so that N $(=N^{\mathrm{T}})$ can be regarded as incorporating the basic finite coupling constants of the gauge fields and $U(\varphi)$ contains those for the scalar fields. For a simple group G, Z_A must be proportional to the unit matrix and in this case we may write $N = 1g^2$. With minimal subtraction Z_A, Z_φ and V_u contain to each order in the loop expansion just poles in ε with coefficients polynomial functions of N, derivatives of U and also, for V_u, φ explicitly.

For the gauge-invariant quantities Z_A and $V_u(\varphi)$ the results of van Damme (1982) and Jack (1983), with the slight generalisations to allow for N in (6), are at one loop

$$\begin{aligned} Z_A^{(1)} &= (1/\varepsilon)(1/16\pi^2)(\tfrac{22}{3}C - \tfrac{1}{3}R), \\ V_u^{(1)} &= (1/\varepsilon)(1/16\pi^2)[\tfrac{1}{2}\operatorname{Tr}(U''^2) + 3U'^{\mathrm{T}}T^2\varphi + \tfrac{3}{2}\operatorname{Tr}(PNPN)]. \end{aligned} \tag{7}$$

Here

$$\begin{aligned} &\operatorname{Tr}(t_a^{\mathrm{ad}} t_b^{\mathrm{ad}}) = -C_{ab}, &&\operatorname{Tr}(T_aT_b) = -R_{ab}, \\ &T^2 = N_{ab}T_aT_b, &&P_{ab}(\varphi) = -\varphi^{\mathrm{T}}T_aT_b\varphi, \end{aligned} \tag{8}$$

and U_i', U_{ij}'' denote the obvious derivatives with respect to φ of U. At two loops

$$\begin{aligned} Z_{A,ab}^{(2)} &= (1/\varepsilon)(16\pi^2)^{-2}[\tfrac{34}{3}(C^2N)_{ab} - \tfrac{1}{3}(RCN)_{ab} - 2\operatorname{Tr}(T^2T_aT_b)], \\ V_u^{(2)} &= (1/\varepsilon^2)(32\pi^2)^{-2}[\![-27\varepsilon N_{ab}\operatorname{Tr}(t_a^{\mathrm{ad}}NPt_b^{\mathrm{ad}}NP) \\ &\quad + \operatorname{Tr}\{[(-44+\tfrac{161}{6}\varepsilon)C + (2-\tfrac{7}{3}\varepsilon)R]NPNPN\} \\ &\quad - (36+30\varepsilon)(NPN)_{ab}\varphi^{\mathrm{T}}T_aT_bT^2\varphi \\ &\quad + (24+2\varepsilon)N_{a'a}N_{b'b}\varphi^{\mathrm{T}}T_{a'}T_{b'}U''T_bT_a\varphi \\ &\quad + 18\varphi^{\mathrm{T}}T^2U''T^2\varphi + (18+3\varepsilon)U'^{\mathrm{T}}T^2T^2\varphi \\ &\quad + [(-38+\tfrac{143}{6}\varepsilon)NCN + (2-\tfrac{11}{6}\varepsilon)NRN]_{ab}U'^{\mathrm{T}}T_aT_b\varphi \\ &\quad - (12+10\varepsilon)(NPN)_{ab}\operatorname{Tr}(U''T_aT_b) \\ &\quad + (12-2\varepsilon)\operatorname{Tr}(U''^2T^2) - 6\varepsilon N_{ab}\operatorname{Tr}(U''T_aU''T_b) \\ &\quad + 12U_{ij}''U_{ijk}'''(T^2\varphi)_k + 2U_{ij}''U_{ijkl}''''U_{kl}'' \\ &\quad + (2-\varepsilon)U_{ij}''U_{ikl}'''U_{jkl}''' + \tfrac{1}{6}\varepsilon U'^{\mathrm{T}}S\varphi]\!], \end{aligned} \tag{9}$$

with $S_{ij} = U_{iklm}''''U_{jklm}''''$. It is easy to see that $NC = CN = -N_{ab}t_a^{\mathrm{ad}}t_b^{\mathrm{ad}}$ commutes with R.

To set up renormalisation group β functions we relate Z_A, V_u to unrenormalised, cut-off or ε-dependent, coupling parameters by

$$Z_A\mu^{-\varepsilon} = N_0^{-1}, \qquad \mu^{-\varepsilon}V_u(\mu^{\varepsilon/2}\varphi) = U_0(\varphi). \tag{10}$$

N_0, U_0 are required to be independent of μ and differentiating then gives

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_A=\varepsilon Z_A,\qquad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}V_u(\varphi)=-\varepsilon\left(\tfrac{1}{2}\varphi_i\frac{\partial}{\partial\varphi_i}-1\right)V_u(\varphi). \tag{11}$$

Arising from (10) N, U have an induced dependence on μ and the corresponding β functions are therefore defined by

$$\hat{\beta}_N=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}N\bigg|_{N_0,U_0},\qquad \hat{\beta}_U=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}U\bigg|_{N_0,U_0,\varphi}. \tag{12}$$

To zeroth order in the loop expansion from (6) and (11), we get

$$\hat{\beta}_N^{(0)}=-\varepsilon N,\qquad \hat{\beta}_U^{(0)}=-\varepsilon B,\qquad B(\varphi)=\left(\tfrac{1}{2}\varphi_i\frac{\partial}{\partial\varphi_i}-1\right)U(\varphi). \tag{13}$$

If we now let

$$\hat{\beta}_N=-\varepsilon N+\beta_N,\qquad \hat{\beta}_U=-\varepsilon B+\beta_U, \tag{14}$$

then, regarding Z_A, V_u as depending on N, U (11) can be recast as

$$\begin{aligned}&\beta_N\frac{\partial}{\partial N}Z_A+\beta_U\frac{\partial}{\partial U}Z_A=\varepsilon\left(1+N\frac{\partial}{\partial N}+B\frac{\partial}{\partial U}\right)Z_A,\\ &\beta_U\frac{\partial}{\partial U}V_u+\beta_N\frac{\partial}{\partial N}V_u=\varepsilon\left(1+N\frac{\partial}{\partial N}+D\right)V_u,\\ &D=B\partial/\partial U=\tfrac{1}{2}\varphi_i\partial/\partial\varphi_i.\end{aligned} \tag{15}$$

By virtue of renormalisability β_N, β_U are finite, independent of ε. These equations may straightforwardly be solved iteratively in the loop expansion. To two loops Z_A is independent of U and also, since $Z_A^{(1)}$ is independent of N as well, we get

$$-N^{-1}\beta_N^{(1)}N^{-1}=\varepsilon Z_A^{(1)},\qquad -N^{-1}\beta_N^{(2)}N^{-1}=\varepsilon(1+N\partial/\partial N)Z_A^{(2)}=2\varepsilon Z_A^{(2)}. \tag{16}$$

Further, for β_U

$$\begin{aligned}&\beta_U^{(1)}=\varepsilon(1+N\partial/\partial N+D)V_u^{(1)},\\ &\beta_U^{(2)}=\varepsilon\left(1+N\frac{\partial}{\partial N}+D\right)V_u^{(2)}-\beta_U^{(1)}\frac{\partial}{\partial U}V_u^{(1)}-\beta_N^{(1)}\frac{\partial}{\partial N}V_u^{(1)}.\end{aligned} \tag{17}$$

Since V_U is polynomial in U the derivatives with respect to U are well defined. Thus, for instance

$$B\partial U_i'/\partial U=B_i'$$

and from (13) and (15) the action of the derivative operator D is given by

$$\begin{aligned}&DU_i'=-\tfrac{1}{2}U_i',\qquad DU_{ij}''=0,\qquad DU_{ijk}'''=\tfrac{1}{2}U_{ijk}''',\qquad DU_{ijkl}''''=U_{ijkl}'''',\\ &(N\partial/\partial N+D)V_u^{(n)}=(n-1)V_u^{(n)},\qquad n=1,2.\end{aligned} \tag{18}$$

Hence with (7) and (9), (16) gives to two-loop order

$$\beta_N=-(16\pi^2)^{-1}N(\tfrac{22}{3}C-\tfrac{1}{3}R)N-(16\pi^2)^{-2}N(\tfrac{68}{3}C^2N-\tfrac{2}{3}RCN-4R')N, \tag{19}$$

for $R'_{ab} = \mathrm{Tr}(T^2T_aT_b)$ and from (17), using (18),

$$\beta_U = (16\pi^2)^{-1}\{\tfrac{1}{2}\mathrm{Tr}(U''^2) + 3U'^{\mathrm{T}}T^2\varphi + \tfrac{3}{2}\mathrm{Tr}(PNPN)\}$$
$$+ (16\pi^2)^{-2}\{-\tfrac{27}{2}N_{ab}\,\mathrm{Tr}(t_a^{\mathrm{ad}}NPt_b^{\mathrm{ad}}NP)$$
$$+\tfrac{7}{6}\mathrm{Tr}[(\tfrac{23}{2}C-R)NPNPN] - 15(NPN)_{ab}\varphi^{\mathrm{T}}T_aT_bT^2\varphi$$
$$+N_{a'a}N_{b'b}\varphi^{\mathrm{T}}T_{a'}T_{b'}U''T_bT_a\varphi + \tfrac{3}{2}U'^{\mathrm{T}}T^2T^2\varphi$$
$$+\tfrac{11}{12}(13NCN - NRN)_{ab}U'^{\mathrm{T}}T_aT_b\varphi - 5(NPN)_{ab}\,\mathrm{Tr}(U''T_aT_b)$$
$$-\mathrm{Tr}(U''^2T^2) - 3N_{ab}\,\mathrm{Tr}(U''T_aU''T_b)$$
$$+\tfrac{1}{12}U'^{\mathrm{T}}S\varphi - \tfrac{1}{2}U'''_{ikl}U''_{ij}U'''_{jkl}\}. \tag{20}$$

It is a non-trivial check that the $1/\varepsilon$ terms on the right-hand side of the expression for $\beta_U^{(2)}$ in (17) cancel. This is straightforward apart from the requirement

$$(NCN)_{ab}U'^{\mathrm{T}}T_aT_b\varphi + 2N_{a'a}N_{b'b}\varphi^{\mathrm{T}}T_{a'}T_{b'}U''[T_b, T_a]\varphi = 0$$

which is true by virtue of $\mathrm{Tr}(t_c^{\mathrm{ad}}Nt_d^{\mathrm{ad}}N) = -(NCN)_{cd}$ and an identity coming from the gauge invariance of U given by Jack (1983).

The result (19) subsumes (for no fermions) that of Jones (1982) for a gauge group of the form $G_1\times G_2$, with G_1, G_2 simple. In this case φ is expressible as a product of representations of G_1, G_2 with the generators of G of the form $T = T_1 + 1_2 + 1_1\times T_2$. If d_1, d_2 are the dimensions of the two representations for T_1, T_2 then we have

$$N = \begin{pmatrix} g_1^2 & 0 \\ 0 & g_2^2 \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \qquad R = \begin{pmatrix} R_1d_2 & 0 \\ 0 & R_2d_1 \end{pmatrix}. \tag{21}$$

Further, if the two representations are irreducible, $T_1^2 = -C_{T_1}1_1$, $T_2^2 = -C_{T_2}1_2$ we get $R' = (C_{T_1}g_1^2 + C_{T_2}g_2^2)$. Hence β_{g_1}, β_{g_2} can be read off from (19).

As an example of the application of (20) we consider the Higgs potential for the Weinberg–Salam model. With gauge group $SU(2)_T\times U(1)_Y$ the Higgs scalar H is a complex $T=\frac{1}{2}$, $Y=\frac{1}{2}$ doublet with interactions described by the basic Lagrangian density

$$\mathscr{L}_H^{(0)} = (D_\mu H)^\dagger D^\mu H - \tfrac{1}{2}\lambda(H^\dagger H - \tfrac{1}{2}f^2)^2.$$

To apply (20) H is decomposed into real fields, with

$$N = \begin{pmatrix} g^2 1 & 0 \\ 0 & g'^2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = \frac{1}{2}\begin{pmatrix} H^\dagger H 1 & H^\dagger \boldsymbol{\sigma} H \\ H^\dagger \boldsymbol{\sigma} H & H^\dagger H \end{pmatrix},$$

so that

$$\beta_U = (16\pi^2)^{-1}\{(H^\dagger H)^2 3[2\lambda^2 - \tfrac{1}{2}\lambda(3g^2+g'^2) + \tfrac{1}{8}(3g^4 + 2g^2g'^2 + g'^4)]$$
$$+ H^\dagger H f^2 3(\tfrac{1}{4}\lambda(3g^2+g'^2) - \lambda^2) + \tfrac{1}{2}\lambda^2 f^4\}$$
$$+ (16\pi^2)^{-2}\{(H^\dagger H)^2[\tfrac{497}{16}g^6 - \tfrac{97}{48}g^4g'^2 - \tfrac{239}{48}g^2g'^4 - \tfrac{59}{48}g'^6 - \tfrac{313}{16}\lambda g^4 + \tfrac{39}{8}\lambda g^2g'^2$$
$$+\tfrac{229}{48}\lambda g'^4 + 9\lambda^2(3g^2+g'^2) - 39\lambda^3] - H^\dagger H f^2(-\tfrac{385}{32}\lambda g^4 + \tfrac{15}{16}\lambda g^2g'^2 + \tfrac{157}{96}\lambda g'^4$$
$$+6\lambda^2(3g^2+g'^2) - \tfrac{15}{2}\lambda^3) + f^4\lambda^2(3g^2+g'^2)\}. \tag{22}$$

To obtain completely general expressions for arbitrary renormalisable field theories it remains only necessary to consider the contribution of fermions. This will be done elsewhere.

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